Derivation of an Equation for the Cut-Off Diameter of a Frustum-Shaped Cavity at a Specific Resonant Frequency

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Abstract

The behavior of cutoff frequencies in frustum-shaped cavities is necessarily different from the behavior of cutoff frequencies in a cylindrical or rectangular waveguide. The reason for this is that, in the TM mode, the electrical field is not tangential to the conductor that establishes the sides of the cavity, so the assumptions in the derivation of the equations predicting the cutoff frequency in a cylindrical cavity do not hold. This paper derives an equation solvable by systems capable of numerical solutions, like R or Matlab, that predicts the cut-off diameter of a frustum-shaped cavity resonating in the TM mode.

1 Statement of the Problem

In the process of investigating the EMDrive phenomenon, I have been trying to understand cutoff frequencies in waveguides. The most useful form of the EMDrive thrust equation is eq 7 from Roger Shawyer’s Theory Paper:

\[ T = \frac{2P_o}{c} \left( \frac{\lambda_0}{\lambda_{g1}} - \frac{\lambda_0}{\lambda_{g2}} \right) \]

This says that the efficiency converting the power in the first term to thrust is determined by the difference in the second term; I will call this second term the "efficiency term". The first term in the efficiency term, which is determined by the size of the large plate, sets the maximum possible efficiency and the second term of the efficiency term determines the efficiency penalty invoked by being larger than the cutoff diameter for the resonating frequency. \( \lambda_0 \) is the wavelength of the microwave in free space, and \( \lambda_{g} \) is the wavelength of the wave in a waveguide, which is usually constant in any device because most resonant
cavities are either boxes or cylinders. $\lambda_g$ must be longer than the wavelength in free space.

The first term is 100% when the large plate is the size of the universe. Fortunately, it does not drop off quickly and by 10 cm at 2.45GHz, it is still approximately 65-70%. The second term is much more sensitive; it can lose more than 35% efficiency in the first few millimeters. The goal is that it must be very close to small enough that the waveguide cannot transmit this frequency, without being smaller than that cut-off.

![Efficiency penalties of the drive as the plates change sizes](image)

Figure 1: This shows the degradation in efficiency as the large plate and the small plate at the ends of a frustum-shaped cavity, according to the equation proposed by Roger Shawyer. The greater slope of the small plate curve indicates the greater sensitivity the thrust has to imperfections in the small plate dimensions than in the large plate.

Feynman describes guide wavelength with this intuition: while the "speed of light" does not change inside a waveguide, there is an interference pattern in the EM wave between the apparent wave coming from the reflection of the emitter in one wall, and the apparent wave coming from the reflection of the emitter in the other wall. This creates an interference pattern that travels in the direction of propagation where the frequency is the same, but the wavelength is longer, and so the phase velocity of the waves is faster than the speed of light. However, there is a limit when the guide wavelength hits infinity and the waveguide cannot transmit waves with lower frequencies than that.

The image below in Figure 3 is the cavity that Eagleworks built in their
Figure 2: Example of the difference between phase velocity and group velocity. The red dot is traveling at the phase velocity; the green dot is traveling at the group velocity. The group velocity is limited by the speed of light, but the phase velocity in a waveguide is not. If your PDF viewer does not support animated images, view this at the source on Wikipedia.

A recent paper showing thrust generated by their test article. I have labeled the dimensions that are important to this conversation. Note the dotted line above the base of the cavity; this is the cutoff diameter as calculated using the traditional equation for a cylinder, coincident with the walls, at the mode and frequency that they were using. That the cavity small plate diameter is smaller than the dotted line should be a problem if the equations for a cylindrical cavity applied to frustum-shaped cavities.

Figure 3: This drawing shows the cross-section of the cavity designed and used by the Eagleworks experiments in their recent seminal paper. The dotted line at the bottom of the frustum indicates the cut-off diameter of the cavity using the equations derived for a cylindrical cavity.

The trouble is that Feynmann’s intuition does not work in the situation of...
a frustum-shaped cavity resonating in a TM mode (the electric field is along longitudinal axis and the magnetic fields encircle the electric fields in the cross-sectional plane) because the "reflections" of the emitter will shift due to the angle of the walls. It is less that this is a surprising outcome, but an equation for this is not readily available, as, in the words of the Eagleworks paper, ”there are no analytical solutions for the resonant modes of a truncated cone”.

2 Derivation

The derivation of the cutoff frequency, and by extension the cutoff diameter at a given frequency, is given in the Balanis paper on circular waveguides:

\[(f_c)_{mn} = \frac{\kappa'}{2\pi a \sqrt{\epsilon \mu}} \] (Balanis 12b)

The problem is in assumption (5a). Because in the TM mode of a frustum, the electric field is not tangential to the conductor/sides, the electric field along the walls of the cavity need not be 0; it is proportional to \(\cos(\theta)\) where \(\theta\) is the angle between the field and the normal vector to the cavity wall. However, the proof in the Balanis paper can jump from (9) to (10) because the only term that matters is \(J_m'(\beta \alpha)\) as that is the only term that can be 0 in all \(\phi_s\) at once, so it must be 0 at the perimeter where \(\rho = a\). If I cannot assume that the electric field is 0, then I can’t use \(\chi_{mn}\) to get \(d_c\).

Starting the derivation over, the TM mode electric field equation is:

\[E^+_z = -jB_{mn} \frac{\beta^2}{\omega \epsilon \mu} J_m(\beta \rho)[C_2 \cos(m\phi) + D_2 \sin(m\phi)]e^{-j\beta z} \] (Balanis 23)

On the assumption that we are at the cutoff, then:

\[\beta_z = 0 \text{ and } \beta = \beta_\rho = \omega \sqrt{\mu \epsilon} \] (Balanis 12a)

\[\Leftrightarrow \begin{cases} \text{Substitution} \\ \text{Multiplication by 0} \end{cases} \]

\[E^+_z = -jB_{mn} \frac{\beta^2}{\omega \epsilon \mu} J_m(\beta \rho)[C_2 \cos(m\phi) + D_2 \sin(m\phi)]e^0 \]

\[\Leftrightarrow \begin{cases} \text{Anything to the exponent of 0 is 1} \end{cases} \]

\[E^+_z = -jB_{mn} \frac{\beta^2}{\omega \epsilon \mu} J_m(\beta \rho)[C_2 \cos(m\phi) + D_2 \sin(m\phi)]\]

Note that this is not the section for TM modes; in the TM section it says the derivation is basically the same and they skip a bunch of steps

\[\text{https://forum.nasaspacelight.com/index.php?action=dlattach;topic=37642.0;attach=1049930} \]
If we do not assume that the electric field $= 0$ at the wall of the cavity, then the electric field in a cross-section of the cavity projects a wave form along the wall where the electric field at a lateral node, dictated by $m$, is 0, and the electric field elsewhere is $cos(\theta)$ of the maximum electric field along the same angle $\phi$ across the cross-section (This part I’m not totally sure about; I’m not 100% sure it is $cos(\theta)$ of the $n$th node; it could be $cos(\theta)$ of some other value, but I don’t think so). The local maximum of the electric field near the wall along any given angle $\phi$ is when:

$$\chi_{mn}' = \beta_{\mu} \rho$$

$$\Leftrightarrow \langle \text{ Substitution } \rangle$$

$$E_{z}^{+} = -j B_{mn} \omega \frac{\mu_{e}}{\omega_{T}} J_{m}(\beta_{\mu} \rho)[C_{2} cos(\mu \phi) + D_{2} sin(\mu \phi)]$$

$$\Leftrightarrow \langle \text{ Simplification of the fraction } \rangle$$

$$E_{z}^{+} = -j B_{mn} \omega J_{m}(\beta_{\mu} \rho)[C_{2} cos(\mu \phi) + D_{2} sin(\mu \phi)]$$

Note that we use $\chi'$, not $\chi$ as is normally used in the cutoff equation in TM mode. In those equations, we are looking for the zeros of the equation to enforce a boundary condition at the wall; here we are trying to find the maximum of the $n$th node.

If

$$E_{z}^{+}_{\chi\rho} = cos(\theta)E_{z}^{+}_{\chi_{\max}}$$

$$\Leftrightarrow \langle \text{ Substitution } \rangle$$

$$cos(\theta) = \frac{-j B_{mn} \omega J_{m}(\beta_{\mu} \rho)[C_{2} cos(\mu \phi) + D_{2} sin(\mu \phi)]}{-j B_{mn} \omega J_{m}(\chi_{mn}')[C_{2} cos(\mu \phi) + D_{2} sin(\mu \phi)]}$$

$$\Leftrightarrow \langle \text{ Simplification } \rangle$$

$$cos(\theta) = \frac{J_{m}(\beta_{\mu} \rho)}{J_{m}(\chi_{mn}')}$$

Meanwhile, based on the geometry of the cavity, it is also true (with the understanding that the goal of the design is that $\rho_{c}$ represents the diameter of the small end of the frustum in an optimal design) that:

$$cos(\theta) = \frac{(\frac{1}{2} \text{base} - \rho_{c})}{\sqrt{(\frac{1}{2} \text{base} - \rho_{c})^{2} + \text{height}^{2}}}$$
\[ \frac{\left( \frac{1}{2} \text{base} - \rho_c \right)}{\sqrt{\left( \frac{1}{2} \text{base} - \rho_c \right) + \text{height}^2}} = \frac{J_m(\beta \rho_c)}{J_m(\chi_{mn})} \]

\[ \frac{\left( \frac{1}{2} \text{base} - \rho_c \right)}{\sqrt{\left( \frac{1}{2} \text{base} - \rho_c \right) + \text{height}^2}} = \frac{J_m(\omega \sqrt{\mu \varepsilon \rho_c})}{J_m(\chi_{mn})} \]

\[ \frac{\left( \frac{1}{2} \text{base} - \rho_c \right)}{\sqrt{\left( \frac{1}{2} \text{base} - \rho_c \right) + \text{height}^2}} = \frac{J_m\left(\frac{2\pi f \rho_c}{c}\right)}{J_m(\chi_{mn})} \]

This equation should be solvable numerically to discover \( \rho_c \). Setting it equal to 0 to find a numerical solution, it is:

\[ 0 = \left( \frac{1}{2} \text{base} - \rho_c \right)^2 + \text{height}^2 \frac{J_m\left(\frac{2\pi f \rho_c}{c}\right)}{J_m(\chi_{mn})} - 1 \]

3 Conclusion

The goal of this paper is to provide a tool that can be used to design frustum-shaped resonance cavities. The utility of this in general usage is limited; except for the phenomenon of the EMDrive, frustum-shaped resonance cavities are not generally useful. However, given a dimension for the base diameter of the frustum, a resonant mode, and a desired frequency (which implies a height, as the height must be an integer multiple of the half-wavelength), solving the equation above for \( \rho_c \) will provide the smallest possible diameter of the top plate of the cavity. The solution for this equation with the dimensions of the Eagleworks cavity is included in Figure 3 as the dotted line above the top surface. (This calculation does not incorporate the use of the dielectric used near the top plate.) The fact that the calculated dimension (129.633 mm, in this case) is shorter than the dimension of the small end of the frustum explains why the cavity can support resonance without extreme resistance.

This understanding of the effect of the pitch of the walls of the cavity on the cut-off diameter has an additional meaning. It explains why, as Roger Shawyer explains in his recent patent filing for generation 2 of the EMDrive, a spherical cap on the cavity does not destroy the resonance of the cavity. It would otherwise seem that, as the radius of the cross-section of the spherical cap decreases, the cut-off diameter would surely be violated and resonance in such a shape would not be possible. However, the increasing angle that the surface of the spherical cap makes with the electric field lengthens the cut-off
diameter, making the question increasingly irrelevant.

This further suggests that such a curved cap must not be such that the angle tangent to the curved cap be steeper than the angle of the side. If the diameter of the cavity at the top of the straight-sided section is at the cut-off diameter for that angle, then a steeper angle, represented by the angle of the tangent line to the curved cap, would create an environment where the wave would be traveling beyond the cut-off and the wave equation would collapse destroying the resonance in the cavity. This means that an extremely wide base, creating very shallow walls and therefore a very small cut-off diameter, may not be feasibly used with a curved cap, as such a cap would necessarily be elliptical and distended, arcing high above the top of the frustum and introduce a prohibitive cut-off diameter within the cap.

This understanding of the behavior of frustum-shaped cavities leads to testable predictions that can be verified, such as testing resonance in a cavity violating the guidelines in the paragraphs above, and I am hoping that support for the buildanemdrive.org project will help further this understanding.